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THE MATHEMATICS TEACHER

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ESTHETICS AND MATHEMATICS

BY PROFESSOR DAVID EUGENE SMITH

Teachers College

General Field.—In considering mathematics in relation to the beautiful, the range of possibility is so vast that a brief article like this can hardly be expected even to list the salient points of contact. The field might properly include all that we designate as the fine arts or, to use the more expressive phrase of the French, the beaux arts. Painting, for example, might be considered with reference to the works of that great genius in science, in mathematics, and in art—Leonardo da Vinci. Sculpture might equally well be included because of the mathematical principles employed by that majestic user of ponderous masses, Michelangelo. Architecture might have place with reference to the works of that Oxford professor of mathematics, Sir Christopher Wren, who rebuilt ecclesiastical London; engraving, with reference to that gifted artist of Nürnberg, Albrecht Dürer, who published the first modern work on curves; music, with reference to the fact that it always ranked as a branch of mathematics until the sixteenth century; decoration, with reference to the geometric designs found in all ages and reaching their highest degree of perfection in the works of the Moslems; and literature, with reference to the mathematics of poetry, and the poetry of mathematics. Indeed, we might properly include the beauties of nature, where mathematics plays a part of which we are usually quite unconscious.

The subject is so extensive that it is merely possible to suggest a few of the special lines to which we may find it of advantage to give a little thought as we plan our work from day to day.

Crude Efforts at Recognition.—The relation of mathematics to the fine arts is so evident that there have not been wanting those who give it quite adequate recognition in their teaching.

Indeed, it is probable that every teacher of geometry calls attention to some of the esthetic features of the figures studied, and numerous authors have done the same. Thus we have certain European textbooks which give geometric lace patterns in their geometries for girls and geometric design in their geometries for boys, and make other crude attempts at relating the esthetic to the intuitive phase of the subject. In America we have had the Gothic window phase, the colored pattern work, and other features which often wasted time in demonstrative geometry, but which have value if used within reason in the preliminary intuitive stage.

All these efforts are helpful but they are crude, and perhaps this crudity must be expected from such attempts. It is quite possible that we can hope for little more than a recognition of the esthetic in the daily work before our classes, letting the idea appear casually as occasion offers. Certainly if we should attempt to write a textbook that should bring out all the relations of mathematics to the fine arts and to the beauties of nature, the result would be not only unwieldy but esthetically and educationally unsatisfactory. Just so long as learning continues to be looked upon as an unpleasant task, just so long must the esthetic lose its beauty, especially if too consciously related to the daily routine.

As for the parquetry flooring, the Gothic windows, the Arabic decorative work, and the linoleum patterns in our textbooks on demonstrative geometry, the less said the better. Except with a limited number of enthusiastic teachers, although not in general with the most successful in producing mathematical results, the effect of their introduction has been rather disappointing.

Antiquity of the Subject.—The relation of intuitive geometry to design is prehistoric. Every savage who plaits dried reeds for a tent cloth makes use of geometric design. Every potter on the banks of the Tigris or the Nile, long before the era of writing, and still longer before geometry as a science was born, used symmetric forms in his designs or drew such forms upon the wet clay before baking it in the sun.

As civilization advanced, axial symmetry in design gave place to the Golden Section,—the dividing of a line-segment in extreme and mean ratio. The ratio of the shorter of the parts to the longer is about 0.618, or roughly $3/5$, $5/8$, or $8/13$. The division is so pleasing to the eye that it was commonly made the basis of Greek design in vases, in sculpture, and in architecture. In general, a Greek temple has its width to its length in about the ratio 3 : 5, and this ratio will be found in various other features of the building. Such was the Greek taste for agreeable form that this ratio was adopted in later times without any apparent knowledge of its basis; indeed, there is not the slightest direct evidence to show that the Greeks themselves selected the division because of their knowledge of the theory involved. Nature had already chosen it hundreds of thousands of years earlier, and had used it in some of her most beautiful forms.

Phyllotaxis.—When we consider the leaves of a tree, or for that matter of any vegetable form, we see that the arrangement is not one of chance. The form of the leaf will vary, and so will its size, but one principle will not vary, namely, that the leaves will so arrange themselves about the stem, the stalk, or the branch as to give a minimum amount of superposition of one upon another, and a maximum exposure of each to the life-giving sun and rain. Nature discovered this necessity millions of years ago. If we accept a familiar theory of the biologist, the plant that did not adopt this law soon perished in the struggle for existence, so that the law became a race habit, as fixed a feature of heredity as the protecting bark of the oak or the eyes of the human being.

Seven or eight centuries ago seems to us a long time, but it is only as part of to-day in comparison with the remoteness of the period when plant habits began to be formed in the ages before the human race appeared. It was less than eight centuries back that Leonardo of Pisa (Leonardo Fibonacci) called attention to an interesting series which we may now write as

$$1, 1, 2, 3, 5, 8, 13, 21, 34, \cdots,$$

in which the sum of any two consecutive terms is equal to the term immediately following; that is, if u_n is the nth term, then

$$u_{n-1} + u_n = u_{n+1}$$

The ratio of the consecutive terms, beginning with the first pair, are $1/2$, $2/3$, $3/5$, $5/8$ $8/13$ $13/21$, $21/34$, $\cdots$ and as the series proceeds, this ratio approaches the limit $\frac{1}{2}(\sqrt{5}-1)$, or approximately 0.618. Thus, $\frac{1}{2} < 0.618$, $\frac{2}{3} > 0.618$, $\frac{3}{5} < 0.618$, $\frac{5}{8} > 0.618$, and so on, but $u_n : u_{n+1} \rightarrow \frac{1}{2}(\sqrt{5} - 1)$ as $n \rightarrow \infty$.

Now one curious fact is that the division of a line in extreme and mean ratio gives us, for the two segments, $\frac{1}{2}(\sqrt{5} - 1) = 0.618$, and $1 - \frac{1}{2}(\sqrt{5} - 1) = \frac{3}{2} - \frac{1}{2}\sqrt{5} = \frac{1}{2}(3 - \sqrt{5}) = 0.382$. Therefore the limit of the Fibonacci series gives us the law of the Golden Section, the *Sectio Aurea* or the *Divina Proportio* of the early Latin books of the Renaissance.

But how is all this connected with the leaves on a tree? We are not at all certain; but there are those who affirm that the ideal angle for many plant leaves is $\frac{1}{2}(3 - \sqrt{5}) \cdot 360°$, 137.52°, or approximately 137° 31′, and that higher plants tend to make use of this angle, that is, of the Golden Section division of 360°, in order to secure the greatest amount of light and rain for their leaves.

Furthermore, it seems to be reasonably safe to assert that if we take a variety of plants and examine their leaves, applying the dividers to the measurement of the distances between the ribs, or between their branch points on the stem, or between the branch points of the flower stems, we shall find the Golden Section more often than one would suspect; that is, such divisions as 1 : 2, 2 : 3, 3 : 5, and 5 : 8. If we wish to save ourselves the trouble of measurement, we may examine the careful drawings and the measurements in such books as Pfeiffer's *Der Goldene Schnitt*, where the work has been done with great attention to accuracy and where the conclusions seem perfectly reasonable.

So we see that Nature seems to have discovered something of the esthetic and economical features of the law of the Golden Section millions of years before Fibonacci gave to the world the series which bears his name, and this series was known six centuries before the law of phyllotaxis, of leaf arrangement, was suggested; but that the eye of the artist had perceived the beauty of this division back in the early centuries of civilization.

Rise of the Golden Section in Art. How came the artist to appreciate the beauty of the golden cutting of a line, of the measuring of two lines so that their ratio should be 3 : 5 or 5 : 8 in preference to the more primitive 1 : 1? Nature changes our habits slowly. In our childhood we admire a fern leaf for its symmetry, for the unit ratio of one half to the other half. It is only in mature years that we notice the more refined ratio of lengths of segments of the stalk or the branches, and observe the approximate ratio 2:3, 3:5, or 5:8. The race has grown in the same way; and it was probably from the observation of natural forms, rather than from the Pythagorean use of the Golden Section in the construction of the regular pentagon, that it came to recognize the beauty of the division of which we have been speaking.

When Pacioli, early in the sixteenth century, wrote upon the subject he adopted a name then current and spoke of the Divine Ratio, *Divina Proportio,* and the term has more significance than one would believe until he has given considerable thought to the subject.

Spirals.—Just as we find, if we seek them, unthought-of beauties in elementary geometry, unthought-of connections between the simple propositions of the high school and the common forms in which Nature delights, so we find in college mathematics similar beauties and relationships if only we look for them.

Consider, for example, the universality of the spiral. We speak of Archimedes as the first to study this form, and as to scientific study it is probably true that it is due to him or to his friend Conon; but ages before Archimedes and Conon lived the spiral was a favorite decoration in art; and millions of years before man recognized its beauties Nature was making constant use of it in her varied forms. It is not merely that the spiral is found in eocene tertiary foraminifers, in the polyzoans of the Torres Straits, in the glass sponge, in the Nautilus, in sheeps' horns, in crystals of sulphur, or in the nebulas in which we see great solar systems or great universes in the making; these evidences are interesting, but what is more significant is that certain of these forms coincide to a notable degree of accuracy, with the logarithmic spiral, while others evidently inspired those artists who designed the beautiful curve of the Ionic capital. Not only is the spiral one of the great cosmic forms, but it is also one of the evidences of a great biologic law and of a great esthetic law as well.

Newton showed that if attraction had varied inversely as the cube of the distance instead of the square, the heavenly bodies would revolve in logarithmic spirals rushing with increasing rapidity out into infinite space. May this not suggest that in some way the inverse square is the law of attraction and the inverse cube the law of production? It is so in the genesis of solar systems, it is so in the growth of thousands of animal and vegetable forms—but is it a universal law? No one knows; we can only say that a mathematical law seems here to show itself in life—in the life of worlds, in the life of the world's fauna and flora, and in the life of mankind as well. With what beautiful unconsciousness of this possible law did Holmes touch upon the sublime fact in his poem on the Nautilus:

"Build thou more stately mansions, O my soul!"

That poet of science, Henri Fabre, has called attention to the web in which the Epeira weaves her logarithmic spiral so skillfully that "one would believe her to be thoroughly versed in the laws" of the curve itself. He asks, as he watches the spider's labors:

"Now is this logarithmic spiral, with its curious properties, merely a conception of the geometers, combining number and extent, at will, so as to imagine a tenebrous abyss wherein to practice their analytical methods afterwards? Is it a mere dream in the night of the intricate, an abstract riddle flung out for our understanding to browse upon?"

In reply he proceeds:

"Let us study, in this connection, the Ammonites, those venerable relics of what was once the highest expression of living things, at the time when the solid land was taking shape from the oceanic ooze. Cut and polished lengthwise, the fossil shows a magnificent logarithmic spiral, the general pattern of the dwelling which was a pearl palace, with numerous chambers traversed by a siphuncular corridor.

"To this day, the last representative of the Cephalopoda with partitioned shells, the Nautilus of the Southern Seas, remains faithful to the ancient design; it has not improved upon its distant predecessors. It has altered the position of the siphuncle, has placed it in the center instead of leaving it on the back, but it still whirls its spiral logarithmically as did the Ammonites in the earliest ages of the world's existence."

He also has these inspiring words relating to the science of which the logarithmic curve is only a single feature:

"Geometry, that is to say, the science of harmony in space, presides over everything. We find it in the arrangement of the scales of a fir-cone, as in the arrangement of an Epeira's lime-snare; we find it in the spiral of a Snail-shell, in the chaplet of a Spider's thread, as in the orbit of a planet; it is everywhere, as perfect in the world of atoms as in the world of immensities."

Has all this anything to do with the esthetic—with the arts of man? Does this very question suggest the error that esthetics must be confined to the products of the human hand? Sir Thomas Browne tells us better, in his *Religio Medici,* when he says, "All things are artificial; for nature is the art of God."[1] And Victor Hugo tells us better still when he says:

"Mathematics plays its part in art as well as in science. There is algebra in astronomy, and astronomy is akin to poetry; there is algebra in music, and music is also akin to poetry."

Henri Fabre commented upon these words by adding:

"Algebra, the poem of order, has magnificent flights. I look upon its formulas, its strophes, as superb."

Music and Mathematics.—As already stated, music was a branch of mathematics until the sixteenth century. Pythagoras, in the sixth century B.C., had noticed that musical strings of equal length, when stretched by weights in the ratio of $\frac{1}{2} : \frac{2}{3} : \frac{3}{4}$, produced intervals which are an octave, a fifth, and a fourth, and thus he was led to include music among the four mathematical disciplines—arithmetic, geometry, astronomy, music. When Shakespeare wrote,

> "I do present you with a man of mine,
> Cunning in music and the mathematics,"

his words contained no element of surprise, for the union of the two was still recognized by long tradition.

But leaving tradition we might seek for a greater truth in Byron's line that

> "There's music in all things, if men had ears,"

which is so axiomatic that we may say that we shall be found wanting if we fail to bring out from our everyday work in mathematics the music, the rhythm, the uplift of spirit, the harmony that is there. When Longfellow wrote,

[1] On the relation of mathematics to art the student may with profit consult Colman, S., *Nature's Harmonic Unity,* New York, 1912; Cook, T. A., *The Curves of Life,* New York, 1914, and various other works of the same type.

"Music is the universal language of mankind,"

he might have written "mathematics" in place of "music," and perhaps with even greater truth.

Tradition should not easily be pushed aside. Any attempt to discard the experience of the race must be as short lived in art and in education as it is in government and in morals. We may speak of superstitions in social affairs, in science, in education, in religion, and in the belief that mathematics and music belong together; but superstitions are about as often based on truth as on falsehood. Astrology is a superstition, but we are not even yet prepared to say, scientifically speaking, that there is no truth whatever in any part of its doctrine. Inoculation was scoffed at as superstitious for generations before Jenner, and for centuries the "regular" physicians condemned as a superstition the folk-belief that the rat spread the plague. For eighteen hundred years the followers of Galen ridiculed the ancient Babylonian superstition that flies carry disease, and yet every pupil in school to-day knows it as a scientific fact.

All of this may seem like a discontinuity, as we say in mathematics,—a break in our graph representing the relation of mathematics to music. It is, however, not at all a cut in the line; it is merely introduced to show that traditions are not mere superstitions, and superstitions are not always wrong. The world believed in ghosts; then came the theorists in science and the ghost was banished; now comes the Psychical Research Society, with a worthy following of scientific minds, and the ghost walks once more. Music was mathematics; then comes the theorists and the relation is severed; now comes modern science and proceeds to photograph a sound wave of a violin, and the resulting curve is merely such a graph as any pupil in the high school might either construct or study. A college student may even determine the equation of the curve, and thus link music to mathematics in a way of which Pythagoras never could have dreamed. Indeed, in the theory of sound in general, the energy, or the intensity, varies as the square of the product of amplitude and frequency; that is

$$I = kn^2 A^2,$$

[2] For graphs and equations illustrating this point see D. C. Miller, *The Science of Musical Sounds*, New York, 1916.

or, for purposes of a graph,

$$I = n^2 A^2.$$[2]

If $n = 1$ and $A = 1$, then $I = 1$, and we have a wave-shaped graph of a certain shape. If $n = 1$ and $A = 2$, we have $I = 4$ and a graph with higher peaks. If $n = 2$ and $A = 1$, we have $I = 4$ and the graph has lower peaks, and if $n = 3.3$ and $A = 0.3$, then $I = 1$ and the graph is a mere ripple upon the surface of the water.

Thus we find in these variants of the curve of sines, in these pictures of the "breathings of the sea," a relationship between music and mathematics that for centuries formed a kind of superstition of the race, but is now a simple mathematical fact.

A German writer, a philosopher with the heart of a poet,—Helmholtz,—speaking of the remarkable relation between these two branches of intellectual activity, described them as

"Mathematics and music, the most sharply contrasted fields of scientific activity which can be found, and yet related, supporting each other, as if to show the secret connection which ties together all the activities of our mind."

Perhaps it was this very expression that led Sylvester to say:

"May not Music be described as the Mathematic of sense, Mathematic as Music of the reason? the soul of each is the same! Thus the musician *feels* Mathematic, the mathematician *thinks* Music,—Music the dream, Mathematic the working life—each to receive its consummation from the other when the human intelligence, elevated to its perfect type, shall shine forth glorified in some future Mozart-Dirichlet or Beethoven-Gauss—a union already not indistinctly fore-shadowed in the genius and labours of a Helmholtz!"[3]

Conclusion.—It should be quite unnecessary to say that these remarks upon the scientific relation of mathematics to the fine arts are not made for the purpose of directly influencing the instruction of classes in algebra, in painting, in trigonometry, or in piano playing. In fact, the remarks as a whole are intended to draw us away from the idea that mathematics is a science apart, when it should be looked upon as the science which

[3] For an authoritative article on this subject, see R. C. Archibald, "Mathematicians and Music," *American Mathematical Monthly*, Vol. 31, p. 1.

binds together all the arts of man. Well did Cicero say, "All arts which relate to mankind have a certain common bond,"[4] and whether or not we exaggerate mathematics sufficiently in our minds to make it seem to be this binding energy, at any rate we do well to be so assured of the relation of this science to the esthetic in its varied forms, as to instil into the minds of our pupils the consciousness that such a relationship exists. If we cannot state as an equation the precise nature of this relationship, nor give a precise account of the common characteristics of two disciplines which seem to the unthinking to be far apart, let us consider the words of Lord Balfour, the greatest philosopher among modern statesmen, when speaking of the two great divisions of human emotions:

"Of highest value in the contemplative division is the feeling of beauty; of highest value in the active division is the feeling of love. . . . Love is governed by no abstract principles; it obeys no universal rules. It knows no objective standard. It is obstinately recalcitrant to logic. Why should we be impatient because we can give no account of the characteristics common to all that is beautiful, when we can give no account of the characteristics common to all that is lovable?"

And why should we who dwell in the domain in which Pythagoras ruled, and in which Archimedes held sway in later times—and Descartes, and Fermat, and Leibniz, and Newton—why should we be impatient because we can only feel the bonds that unite mathematics and esthetics, although we are without power to express the law of union?

[4] Omnes artes quae ad humanitatem pertinent habent quoddam commune uinculum.

FUNCTIONS IN GENERAL, AND THE FUNCTION [x] IN PARTICULAR

By PROFESSOR WALTER B. CARVER

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One of the earliest notions encountered in mathematics is that of a function. The word is regarded at first as being synonymous with the word "expression"—a function of x is an expression in x. Thus x^2, $\frac{x^2}{x-1}$, $\sqrt{x}$, $\sqrt{x-4}$ are functions of x. When the number x is given, these expressions indicate operations which can be performed upon the number x to obtain a new number which is the value of the expression. For a given expression there may be some restrictions upon the values that may be given to x if the expression is to have meaning. Thus, for $\frac{x^2}{x-1}$, x can not be given the value 1, because in this case the expression would call for a division by zero, and there is no such operation in mathematics. And if we are restricted to dealing with real numbers (as is the case in many practical applications) x can not be given negative values in the expression $\sqrt{x}$, and only values equal to or greater than 4 in the expression $\sqrt{x-4}$. In all the examples cited, the value of x being given arbitrarily (except for such restrictions as noted) the value of the expression can be found exactly, or approximated as a decimal to any required number of places, by well-known arithmetical procedures. When we consider such expressions as

$$\sqrt[5]{x+1}, \qquad \sin 2x, \qquad \log(1+x), \qquad 2^x,$$

the difficulty of finding (exactly or approximately) the value of the function for a given value of x increases, but the idea is not essentially different. There must be some arithmetical process for approximating $\sin 2x$ when x is given—otherwise whence the tables of sines? The essential notion of a function of x is that when x is given a value the value of the function is fixed.

One now begins to see that the idea of an *expression* is not essential. A mathematical expression in x is a conventional symbolism representing some operation that is to be performed with the number x. One writes $y = x^2$, meaning that the value of y will be found by multiplying x by itself, and y is thus a function of x. If the notation x^2 had never been invented, we could still say that y is to be found by multiplying x by itself, and y would then be a function of x although there would be *no expression* to represent the relationship. If we agree that y is to be twice x whenever x is an integer and equal to x for all other values of x, then y is a function of x even though we can not write that y is equal to any conventional expression in x. All about us are such related quantities which we encounter in our every-day experience, some of the relationships being expressible in conventional mathematical symbols, and others not. To put the matter precisely:

We say that y is a function of x if y and x are so related that when x is given any value (arbitrary except for certain specified restrictions) [1] the value of y is determined.

When we restrict ourselves to real numbers, the familiar type of graphical representation is helpful in the consideration of functions. Giving a value to x, we find the corresponding value of y, and plot the point having the coordinates (x, y) in a rectangular Cartesian system. The totality of all the points that could be so obtained makes up a graph which is useful in studying the functional relationship between y and x. The graph is usually a curve in the simpler cases (though it is not necessarily so),[2] and the graphs corresponding to the more familiar relationships such as $y = x^2$, $y = \sqrt{x}$, and $y = \sin x$ are shown in our elementary textbooks. In this connection the equation $y = 5$ may be regarded as expressing y as a function of x if it is understood to mean that y has the value 5 *for every value of* x, and it is only with this understanding that we can represent this equation graphically by a line parallel to the x axis.

New expressions are being constantly introduced into mathematics, and many symbols are in use in addition to the more familiar ones in our elementary texts. Thus, sgn x (read "signum x") represents a function which has the value 1 for all positive

[1] Thus if $y = \dfrac{x}{x-1}$, the value of y is determined when x is given any value whatever *except* 1; and similarly, for $y = x!$ (factorial x), x can be given any *positive integer* value.

[2] Thus the graph corresponding to $y = x!$ is a set of isolated points, since $x!$ has meaning only for positive integral values of x.

values of x, -1 for all negative values of x, and zero when x is zero; while $\Gamma(x)$ ("gamma x") is a function closely related to $x!$ but defined for all positive values of x.

A function of considerable interest is denoted by the symbol $[x]$, which may be read "bracket x." It is often roughly described as meaning "the greatest integer in x." To be precise, $y = [x]$ means that when x is an integer y is equal to x, and when x is not an integer y is the greatest integer less than x. Thus for $x = 4.2$, $y = 4$; for $x = \sqrt{29}$, $y = 5$; for $x = \pi$, $y = 3$; for $x = -3.3$, $y = -4$; and for $x = 1 - \sqrt{7}$, $y = -2$. The graph of the function consists of disconnected pieces of straight line

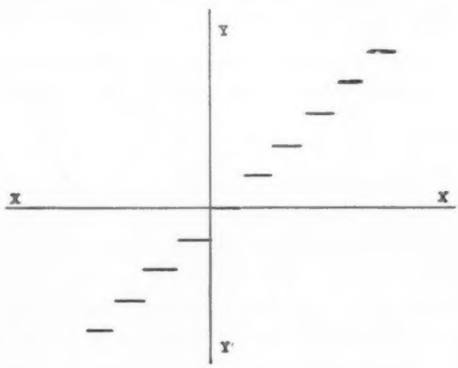

FIG. 1

parallel to the x axis. With regard to this graph it must be understood that the point corresponding to an integral value of x is not at the right hand end of a segment but at the left hand end of the next segment above.

We use this function (though not the expression for it) almost daily. A number of pieces of candy are to be distributed evenly among five boys, any odd pieces left over being given to the dog. If x is the total number of pieces and y the number that the dog gets, what is the equation expressing the relation between y and x? It is a rather curious thing that our ordinary algebraic symbols are not adequate for this purpose. One may readily verify that the relation is

$$y = x - \left[\frac{x}{5}\right].$$

If x represents the weight of a letter in ounces and y the number of cents postage required on it, one might say carelessly that the relation is $y = 2x$; but this is obviously wrong, for if the letter weighs 2.5 ounces one does not pay 5 cents postage. The correct equation is $y = -2[-x]$. If the letter weighs 2 ounces, $[-2] = -2$, and $y = (-2)(-2) = 4$; while if the letter weighs 2.5 ounces, $[-2.5] = -3$, and $y = (-2)(-3) = 6$. If the postmaster has a mean disposition the formula takes the somewhat simpler (but illegal) form $y = 2(1 + [x])$.

If you can buy ice-cream only in quart bricks, and one brick is enough for 8 people, how many bricks will you need to buy for x guests? One thinks at once of the expression $\left[\frac{x}{8}\right] + 1$, and this gives the correct number when x is not a multiple of 8 but gives an unnecessary extra brick when x is a multiple of 8. The correct expression may be seen to be $-\left[-\frac{x}{8}\right]$.

It should be noted that the function $-1-[-x]$ is almost the same function as $[x]$: for values of x other than integers the two expressions are equal; but for integer values of x, $[x] = x$ while $-1-[-x] = x - 1$. The graph for $y = -1 - [-x]$ looks just like that for $y = [x]$, the difference being only that for an integer value of x the point is at the right-hand end of a segment instead of being at the left-hand end of the next segment.

By using this bracket function in combination with the more familiar functions, many interesting and useful relations can be expressed. The reader may find it amusing and profitable to plot the graphs corresponding to such expressions as

$$x - [x], \qquad (x - [x])^2, \qquad x - [x] + (x - [x])^2.$$

Since the function $\frac{x^2 + x + 2}{x^2 + 2}$ has values between 0 and 1 when x is negative, between 1 and 2 when x is positive, and the value 1 when x is zero; it follows that $\left[\frac{x^2 + x + 2}{x^2 + 2}\right]$ has the value 0 when x is negative and the value 1 when x is zero or positive. The graph coincides with the x axis to the left of the origin, then jumps to a point one unit above the origin on the y axis and continues to the right as a line parallel to the x axis and one unit above it. If we replace x by $x - a$ in this function, a being any fixed number, the break in the graph will occur at $x = a$ rather than at the origin. Thus

$$\left[\frac{(x-5)^2 + (x-5) + 2}{(x-5)^2 + 2}\right]$$

has the value 0 when x is less than 5, and the value 1 for x equal to or greater than 5. Evidently the function

$$1 - \left[\frac{(x-a)^2 + (x-a) + 2}{(x-a)^2 + 2}\right]$$

has the value 1 for x less than a and the value 0 for x equal to or greater than a. This enables us to write an equation corresponding to a graph made up of parts of two different curves. Thus

$$y = \left\{1 - \left[\frac{(x-1)^2 + (x-1) + 2}{(x-1)^2 + 2}\right]\right\} x + \left[\frac{(x-1)^2 + (x-1) + 2}{(x-1)^2 + 2}\right]\frac{1}{x}$$

has a graph which consists of the line $y = x$ to the left of the point

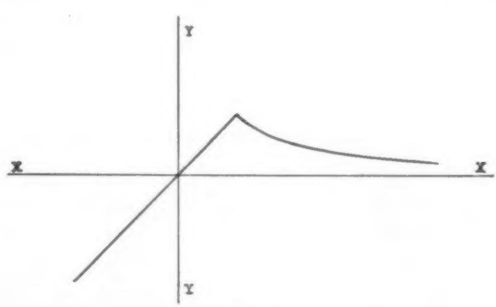

Fig. 2

(1, 1), and of a part of the hyperbola $y = \frac{1}{x}$ to the right of this point.

As a curiosity, the reader may be interested in a parcel post formula. If y represents the required postage on a package, x the weight of the package in pounds, and z the number of the zone,[3] the formula may be written

[3] "Local" delivery corresponds to $z = 0$.

$$y = z + 5 + \left[\left(\frac{6z - 21}{7}\right)\left[\frac{z + 4}{8}\right] + \frac{14 - 4z}{7}\right]$$
$$+ \left\{1 + (2z - 5)\left[\frac{z + 5}{8}\right]\right\}\left\{-1 - \left[-\frac{x + \left[\frac{8 - z}{8}\right]}{1 + \left[\frac{8 - z}{8}\right]}\right]\right\}$$

It may be noted that this is not the only formula that correctly expresses this relation. The reader may be interested in trying to devise a simpler one.

Two arguments may be advanced for the introduction of this function bracket x into our teaching of elementary algebra. First, its meaning is simple,[4] and it expresses a number relation which occurs frequently in our everyday experience and is not expressible in our familiar algebraic symbols. Secondly, it furnishes a simple example of a *discontinuous* function. When a student begins the study of calculus he is not much impressed by a definition of continuity. He has become familiar with a number of mathematical expressions all of which represent functions which are continuous as long as they remain finite; *i.e.*, the only type of discontinuity familiar to him is that exhibited by $\frac{1}{x - 1}$ at $x = 1$. If he gets the idea of continuity at all, he is likely to get the impression that so long as his functions are bounded he need not worry about their continuity. A familiar function, expressed by a familiar notation, that needed watching in this respect would be helpful.

[4] The meaning of $[x]$ is at least as simple as that of x^2 and x^3, and it is certainly much simpler than that of $\sqrt{x}$ or $x^{-2/3}$.

OBJECTIVES IN TEACHING DEMONSTRATIVE GEOMETRY

By PROFESSOR W. D. REEVE
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In THE MATHEMATICS TEACHER for November, 1925 I published an article on "Objectives in the Teaching of Mathematics," a large part of which was a list of specific objectives in elementary algebra. In the March 1927 issue of the same magazine I published a list of objectives to be attained in teaching intermediate algebra. In the preparation of these lists I had the assistance of a large number of my students in Teachers College who are mature teachers of experience. The objectives therein presented have furnished many groups with basic lists of aims which have been used in preparing new courses of study in various parts of the country. In the last two years I have also prepared, with the help of my students a list of objectives to be obtained in the teaching of demonstrative geometry. As was the case with the other two lists, this new group of objectives is not intended to be final, but tentative. We are willing to present them to the readers of THE MATHEMATICS TEACHER because we hope that in this way they will be discussed and some more definite aims established in the teaching of geometry.

In comparison with the great central objectives to be attained in the teaching of demonstrative geometry the specific propositions, principles, definitions, or other geometric facts listed here and which are usually classified as "exercises" are of minor significance. However, we think with facts, and our list seems to us to constitute the mechanical structure around which the course in geometry is built and by means of which the fundamental major objectives are realized.

GENERAL OBJECTIVES

I. To develop an understanding of

  1. The need for proving statements.
  2. The difference between intuitive and demonstrative geometry.
  3. The meaning of

a. axiom. *c.* theorem. *e.* proposition.
b. postulate. *d.* corollary. *f.* converse.

4. Axioms, postulates, and definitions as bases of proof.
5. The various forms of geometric proof as follows:

a. direct. *c.* analytic. *e.* inductive.
b. indirect. *d.* synthetic. *f.* deductive.

6. The various steps in a geometric proof.
7. Statement and proof of the converse of a theorem.
8. Analysis of originals.
9. Geometric construction.
10. Functional relationship.

II. To acquire habits of, and develop power for:
1. Logical thinking.
2. Reasoning.
3. Induction from original problems.
4. Critical attitude.
5. Correct speed.
6. Neatness and accuracy in construction.

III. To develop an appreciation of:
1. "The human worth of rigorous thinking."
2. The practical value of mathematics in life.
3. The aesthetic value of mathematics.

Objectives in Teaching Congruence

I. To develop an understanding of
1. The idea of congruence.
2. The direct method of proof.

II. To extend the knowledge of axioms to include the relation of the whole to its parts.

III. To develop the following abilities:
1. To use the congruence theorems in establishing valid proofs.
2. To select corresponding parts of congruence figures.
3. To pick out overlapping triangles.
4. To use the congruence theorems to measure inaccessible lines.
5. To use the facts regarding an isosceles triangle in establishing proofs.

6. To use the facts regarding an equilateral triangle in a proof.
7. To pick out the significant triangles in a figure.
8. To discover geometric relations in original exercises.

Objectives in Teaching Parallel Lines

I. To develop an understanding of:
   1. Parallel lines.
   2. The postulate of parallels.
   3. Transversal.
   4. The kinds of angles made by a transversal.
      - *a.* Interior.
      - *b.* Exterior.
      - *c.* Alternate.
      - *d.* Corresponding.
   5. The five conditions of parallelism.
   6. The sum of the angles of a triangle.
   7. The relation of an exterior angle of a triangle to the two nonadjacent interior angles.

II. To develop the following abilities:
   1. To draw a line through a given point parallel to a given line.
   2. To pick out equal angles.
   3. To pick out angles whose arms are respectively parallel in the same or in opposite directions.
   4. To use the following in the proofs of subsequent theorems and the solution of original exercises:
      - *a.* The conditions of parallelism.
      - *b.* The fact that the sum of the angles of a triangle is equal to 180°.
      - *c.* The relation between an exterior angle of a triangle and the two nonadjacent interior angles.

Objectives in Teaching Quadrilaterals

I. To establish the following concepts:
   1. Properties of quadrilaterals:
      - *a.* The quadrilateral as a four-sided figure.
      - *b.* The parallelogram as a quadrilateral whose opposite sides are parallel.
      - *c.* The rectangle as an equiangular parallelogram.
      - *d.* The square as an equilateral rectangle.

e. The rhombus as an equilateral parallelogram.
f. The trapezoid as a quadrilateral of which two sides are parallel.
g. The diagonal as joining two non-consecutive vertices.
h. The altitude as the length of the perpendicular between the bases.

2. Distance.
a. Distance between two points as the length of the straight line between them.
b. Distance from a point to a line as the length of the perpendicular dropped from that point to the line.
c. Distance between two parallel lines as the perpendicular distance between them.

II. To develop and establish the following concepts relating to parallelograms:

1. The side relations of parallelograms.
a. Opposite sides equal.
b. Opposite sides parrallel.
c. Two opposite sides equal and parallel.

2. Angle relations of parallelograms.
a. Opposite angles equal.
b. Any two consecutive angles supplementary.

3. Diagonal relations.
a. Diagonals bisect each other.
b. Diagonals divide parallelograms into two congruent triangles.

III. To develop and establish the following concepts relating to parallel lines:

1. Line relations.
a. Two parallel lines are equally distant from each other.
b. Segments of parallel lines cut off by parallel lines are equal.
c. If three or more parallel lines intercept equal segments on one transversal they intercept equal segments on every transversal.

2. Triangle relations.
a. A line parallel to one side of a triangle bisecting another side, bisects the third side.

b. A line joining the midpoints of two sides of triangle, parallel to a third side bisects the third side.

3. Trapezoid relations.

a. A line parallel to the base of a trapezoid bisecting one of the other sides, bisects the opposite side, and is equal to half the sum of the bases.

IV. To develop the ability to analyze originals relating to quadrilaterals.

1. By recognizing equality of line segments.

a. As opposite sides of a parallelogram.

b. As bisected diagonals of a parallelogram.

c. As diagonals of a rectangle or of a square.

d. As segments of parallel lines cut by parallel lines.

e. As line parallel to one side of a triangle or trapezoid which bisects the sides.

2. By recognizing parallelism of line segments.

a. As opposite sides of a parallelogram.

b. As a line which joins midpoints of the sides of a triangle and of a trapezoid.

c. As segments of parallel lines cut by parallel lines.

3. By recognizing equality of angles.

a. As opposite angles of a parallelogram.

b. As angles of a rectangle or of a square.

Objectives in Teaching Polygons

I. To establish the following concepts relating to properties of a polygon:

1. As a rectilinear figure which has three or more sides.
2. Triangle and quadrilateral as polygons with three and four sides respectively.
3. Classification of pentagon, hexagon, and octagon.
4. Regular polygons as polygons which are both equilateral and equiangular.
5. Diagonals as dividing a polygon into triangles.

II. To develop and establish the following concepts:

1. The angle relations of a polygon.

a. The sum of the interior angles of a polygon of n sides is $(n - 2)\ 180°$.

b. Sum of the angles of a triangle and of a rectangle as special cases of a polygon.

c. Each angle of a regular polygon of n sides contains

$$\left(\frac{n-2}{n}\right) 180°.$$

d. The sum of the exterior angles of a polygon is 360°.

III. To develop the ability to analyze originals concerning polygons.

1. By recognizing the angle relations.

a. Sums of angles as depending upon the number of triangles in polygons formed by drawing diagonals from a vertex.

b. Sums of angles of a polygon depending upon the number of sides of a polygon.

c. Sums of angles of a polygon as depending upon the sum of the angles in a triangle.

Objectives in Teaching Geometric Constructions

I. To develop an understanding of:

1. The significance of constructing versus the mere drawing of figures.
2. The development of a synthetic proof by using the analytic method.
3. The terms bisect, trisect, radius, diameter.
4. How to discuss impossible cases of a construction and cases with two solutions.
5. How to check constructions by measuring and by paper folding.

II. To develop the ability to:

1. Draw an arc.
2. Lay off on a line a segment equal to a given line segment.
3. Bisect any angle in any position alone or when it is a part of a figure.
4. Bisect a line segment.
5. Divide a given line into any number of equal segments.

6. Construct a perpendicular—two general cases for position of the point, when line is in any position and when it is a part of a figure.
7. Inscribe a hexagon, a triangle, or a square in a circle.
8. Construct an angle equal to a given angle.
9. To construct a triangle—three cases.
10. Construct a line parallel to a given line.

III. To develop an appreciation of:
1. Practical application of these constructions, including every day uses.
2. Constructions in the realm of the beautiful.

Objectives in Teaching Inequalities

To realize the following objectives:
1. To know the axioms of inequalities.
2. To realize that certain axioms of inequalities are true only for positive quantities.
3. To develop the ability to use the axioms of inequalities in proofs.
4. To develop the ability to prove the following propositions:—
    a. If two sides of a triangle are unequal, the angles opposite these sides are unequal, and the angle opposite the greater side is the greater.
    b. If two angles of a triangle are unequal, the sides opposite these angles are unequal, and the side opposite the greater angle is the greater.
5. To accept the following theorems as postulates:
    a. If two sides of one triangle are equal respectively to two sides of another triangle, but the included angle of the first is greater than the included angle of the second, then the third side of the first is greater than the third side of the second.
    b. If two sides of one triangle are equal respectively to two sides of another triangle, but the third side of the first is greater than the third side of the second, then the angle opposite the third side of the first is greater than the angle opposite the third side of the second.

6. To develop the ability to use the propositions stated above in original exercises.
7. To develop further the ability to use an indirect proof.
8. To develop further the ability to recognize the converse of a proposition.

Objectives in Teaching Circles

I. To develop an understanding of the following concepts:

1. Central angle.
2. Chord.
3. Arc.
   a. Major.
   b. Minor
4. Tangent.
5. Inscribed Angle.
6. Secant.
7. Locus (including the nature of a proof).
8. Incenter.
9. Circumcenter.
10. Orthocenter.
11. Centroid.
12. Postulates.
    a. Those relating to circles.
    b. Those relating to loci.

II. To develop the ability to understand and to apply the following:

1. Through any three points lying in a straight line one circle, and only one, can be drawn. (This should be postulated.)
2. Equal chords of the same circle or of equal circles are equidistant from the center and conversely.
3. If the diameter is perpendicular to a chord, it bisects the chord and its two arcs.
4. An inscribed angle is measured by half its intercepted arc.
5. The locus of a point equidistant from two points is the perpendicular bisector of the line segment joining them.
6. The locus of a point equidistant from two intersecting lines is the pair of lines which bisects the angles formed by the given lines.

III. To understand the content of and be able to apply in the solution of originals the following:

1. Relations between central angles and their arcs, in the same or equal circles.

2. An angle formed by two intersecting chords of a circle is measured by half the sum of the intercepted arcs.
3. An angle formed by two secants of a circle, or by two tangents or by a secant and a tangent, intersecting at a point outside the circle, is measured by half the difference between the intercepted arcs.
4. An angle formed by a tangent and a chord of a circle is measured by half the intercepted arc.
5. Angles inscribed in the same segment, or in equal segments of a circle are equal.
6. An angle inscribed in a semi-circle is a right angle.
7. Relations between chords and their arcs, in the same or in equal circles.
8. If a diameter bisects a chord, which is itself a diameter, it is perpendicular to the chord.
9. If a line is perpendicular to a radius at the end lying on the circle the line is tangent to the circle.
10. If a line is tangent to a circle it is perpendicular to the radius drawn to the point of contact.
11. If tangents to a circle from an external point are drawn they make equal angles with the line joining the given point to the center and their segments from the given point to the points of contact are equal.
12. Relation between lengths of chords and their distances from the center.
13. Relation between arcs of a circle cut off by parallel lines.
14. In the same circle, or in equal circles, two central angles are proportional to their intercepted arcs.
15. The perpendicular bisectors of the sides of a triangle meet in a point.
16. The bisectors of the angles of a triangle meet in a point.
17. The altitudes of a triangle meet in a point.

18. The medians of a triangle intersect in a point which is two thirds of the distance from any vertex to the midpoint of the opposite side.

IV. To develop the ability to construct accurately the following and to perform other constructions based on these problems:

1. To circumscribe a circle about a given triangle.
2. To inscribe a circle in a given triangle.
3. To draw a tangent to a circle through a given point on the circle.
4. To draw a tangent to a circle through a given external point.

Objectives in Teaching Proportion and Similarity

I. To develop an understanding of:

1. The laws of proportion applied to lengths, areas and volumes.

   a. If $\frac{a}{b} = \frac{c}{d}$, then $ad = bc$.

   b. If $ad = bc$, then $\frac{a}{b} = \frac{c}{d}$.

   c. If $\frac{a}{b} = \frac{c}{d}$, then $\frac{a}{c} = \frac{b}{d}$.

   d. If $\frac{a}{b} = \frac{c}{d}$, then $\frac{b}{a} = \frac{d}{c}$.

   e. If $\frac{a}{b} = \frac{c}{d}$, then $\frac{a+b}{b} = \frac{c+d}{d}$.

   f. If $\frac{a}{b} = \frac{c}{d} = \frac{e}{f} = \cdots$,

   then $\frac{a+c+e+\cdots}{b+d+f+\cdots} = \frac{a}{b} = \frac{c}{d} = \cdots$.

2. Vocabulary used in connection with proportional line segments.

   a. Internally and externally proportional.
   b. Mean proportional.
   c. Third proportional.
   d. Fourth proportional.

3. Similarity.
    a. The use of equal ratios in proving two figures similar.
    b. The use of parallel lines in obtaining ratios between line segments.
    c. The conditions necessary to make two triangles similar and also two polygons having more than three sides similar.
    d. The difference between similarity, equality in area, and congruence; and the symbolism relating to these terms.
    e. The use of similar figures to obtain equal ratios between line segments and to prove two angles equal.
    f. The proportions formed by intersecting chords of a circle, the tangent and secant to a circle, the bisector of the angle of a triangle, and the perpendicular from the vertex of the right angle of a right triangle to the hypotenuse.

II. To use the following propositions, either as theorems or as original exercises:

1. If through two sides of a triangle a line is drawn parallel to the third side, it divides the two sides proportionally.
2. Three or more parallel lines cut off proportional line segments on any two intersecting transversals.
3. If a line cuts two sides of a triangle so that the corresponding segments are proportional the line is parallel to the third side.
4. Two mutually equiangular triangles are similar.
5. If two triangles have an angle of one equal to a corresponding angle of the other and the including sides proportional, the triangles are similar.
6. If two triangles have their sides respectively proportional, they are similar.
7. The perpendicular from the vertex of the right angle of a right triangle divides the triangle into two triangles which are similar to each other and to the given triangle.

8. The square of the hypotenuse of a right triangle is equal to the sum of the squares of the other two sides.
9. If two chords of a circle intersect, the product of the segments of either one is equal to the product of the segments of the other.
10. The perimeters of two similar polygons vary directly as any two corresponding sides.
11. If two polygons are similar, they can be separated into the same number of triangles similar to each other and similarly placed.
12. The bisector of an angle of a triangle divides the opposite side into segments proportional to the adjacent sides.
13. If two polygons are composed of the same number of triangles similar each to each and similarly placed the polygons are similar.
14. If from a point outside a circle a secant and a tangent are drawn, the tangent is the mean proportional between the secant and its external segment.

III. To learn how to make the following constructions:
1. Divide a given line segment into parts proportional to any number of given segments.
2. Construct the mean proportional between two given line segments.
3. Construct the fourth proportional to three given line segments.
4. Upon a given line segment corresponding to a given side of a given polygon construct a polygon similar to the given polygon.

Objectives in Teaching Areas of Polygons

I. To understand the meaning of unit of area and equivalent figures.

II. To develop the following abilities:
1. To find the area of a rectangle.
2. To find the area of a parallelogram.
3. To find the area of a triangle.
4. To find the area of a trapezoid.

5. To find the area of an irregular polygon.
6. To prove that the area of a parallelogram is equal to the product of its base by its altitude.
7. To prove that the area of a trapezoid is equal to one-half the product of the altitude by the sum of the bases.
8. To compare the areas of two triangles, or of two rectangles or of two parallelograms, having equal bases and equal altitudes.
9. To compare the areas of two triangles, if they have equal bases but different altitudes, and so for other polygons.
10. To find the ratio of the areas of two similar triangles.
11. To prove that the areas of two similar triangles are to each other as the squares on any two corresponding sides.
12. To find the ratio of the areas of two similar polygons.
13. To prove that the areas of two similar polygons are to each other as the squares on any two corresponding sides.
14. To construct a square equivalent to two given squares.
15. To transform a polygon into an equivalent triangle.
16. To prove geometrically that $(a + b)^2 = a^2 + 2ab + b^2$.
17. To prove geometrically that $(a - b)^2 = a^2 - 2ab + b^2$.
18. To prove geometrically that $(a + b)(a - b) = a^2 - b^2$

Objectives in Teaching the Regular Polygons and the Circle

I. To develop an understanding of the following terms:
1. Center of a polygon.
2. Radius of a polygon.
3. Apothem of a polygon.
4. Inscribed and circumscribed polygons.
5. Sector of circle.
6. Area of circle.
7. Circumference as the length of the circle.

II. To develop the ability to construct the necessary figure and to prove or to show that the figure is the correct one in the following cases:

1. To circumscribe a circle about a regular polygon.
2. To inscribe a circle within a regular polygon.
3. An inscribed equilateral polygon is regular.
4. A circumscribed equiangular polygon is regular.
5. The chords of equal arcs form a regular inscribed polygon.
6. Tangents at the midpoints of equal arcs of a circle form a regular circumscribed polygon.
7. Tangents to a circle at the vertices of the inscribed polygon form a regular circumscribed polygon of same number of sides.
8. To construct a regular polygon of twice the number of sides of a given regular polygon.
9. To inscribe the following figures in a given circle: (*a*) square, (*b*) polygons of $2n$ sides, (*c*) hexagon, (*d*) equilateral triangle.

III. To develop the ability:

1. To prove that the area of a regular polygon is equal to half the product of its apothem by its perimeter.
2. To recognize that the ratio of C to d in a circle is equal to $3.14159\cdots$.
3. To recognize that the circumference of the circle is the limit of the perimeter of the inscribed regular polygon and circumscribed regular polygon.
4. To recognize that the area of a circle is equal to $C \cdot \frac{1}{2}r$.

IV. To develop the ability to use the truths learned in the solution and construction of exercises, more especially concerning:

1. The area of a circle and a polygon.
2. The area of a sector.
3. The finding of any part of a circle with any one part given.
4. The fact that areas of circles vary as the square of d or r.

5. The appreciation of the use of polygons and circles in art, design, and architecture.

V. Possible topics:

Golden section, decagon, pentagon.

Objectives in Teaching "Lines and Planes in Space"

To develop the following abilities:

1. To recognize the limits of the field of plane geometry and to desire to extend this knowledge into three-dimensional space.
2. To develop constructive and spatial imagination.
3. To represent three-dimensional figures.
4. To understand the properties of a plane in space.
5. To learn of the relations of points, lines and planes.
6. To extend the concept of "angle" to include "dihedral angle."
7. To use in the solving of original exercises the theorems concerning the following subjects:
   a. The intersection of two planes.
   b. A perpendicular to intersecting lines.
   c. The perpendiculars to a line.
   d. A perpendicular at a point in a plane.
   e. A perpendicular from a point to a plane.
   f. A perpendicular and obliques.
   g. Two perpendiculars to a plane.
   h. A plane through one of two parallel lines.
   i. Planes perpendicular to a line.
   j. Parallel planes cut by a third plane.
   k. Angles with parallel arms.
   l. A perpendicular to the intersection of planes.
   m. A plane through a perpendicular.
   n. The intersection of perpendicular planes.
   o. A perpendicular to two skew lines.

Objectives in Teaching Metric Solid Geometry

I. To develop the formulas for finding the areas of a:

1. Prism.
2. Regular pyramid.
3. Frustum of a regular pyramid.
4. Curved surface of a circular cylinder.

5. Curved surface of a cone of revolution.
6. Curved surface of a frustum of a cone of revolution.

II. To develop an understanding:
1. Of sections of a prism made by parallel planes.
2. That an oblique prism is equivalent to a right prism under certain conditions.
3. That sections of two pyramids are equivalent under certain conditions.
4. That triangular pyramids are equivalent under certain conditions.
5. That the number of edges of a polyhedron is two less than the sum of the number of vertices and the number of faces.
6. Of the formula for the area of a sphere.

III. To develop the formulas for finding the volume of a:
1. Rectangular perallelopiped.
2. Any parallelopiped.
3. Triangular prism.
4. Any prism.
5. Triangular pyramid.
6. Any pyramid.
7. Circular cylinder.
8. Circular cone.
9. Sphere.

IV. To develop the ability to compute the areas and volumes of the geometric magnitudes mentioned above, *i.e.*, to develop ability to use the derived formulas.

OBJECTIVES IN INTERMEDIATE ALGEBRA

By JOSEPH A. NYBERG

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In the MATHEMATICS TEACHER for March, 1927, Professor Reeve has presented an interesting and useful list of the objectives in intermediate algebra. Every teacher will find in the list the particular items that the teacher believes should be included. When a list is as comprehensive as this one, certain practical questions arise: Can all the objectives be reached? Which objectives can be slighted if necessary, and which are most important? Since Professor Reeve suggests that the list serve as a basis for discussion, I present one answer to these questions, an answer based on ten years' experience with about 500 pupils in classes averaging 37 each.

I assume that the "intermediate" algebra is the pupil's *third* semester of algebra, not the fourth, as it would be if the last half of the eighth grade is devoted to algebra. Also, I assume that this work comes after a year's work in geometry has intervened so that the pupil has not retained all his previous algebraic technique. Semesters vary in length. Ours is twenty weeks, but I allow only 89 days because holidays may take five days or more and some other days are set aside for final examinations. Hence the following outline is in the form of 89 lessons in algebra, each lesson covering one day's work.

1. A rapid survey of the opening chapter in the text. This chapter is always a summary of the fundamental rules and operations. We discuss the evaluation of algebraic expressions, rules of signs, removal and insertion of parentheses, multiplication and division of polynomials.

2. The axioms used in solving linear equations with integral and fractional coefficients. Discussion of transposition, "clearing of fractions," and various typical errors.

3. Drill on linear equations.

4. Solution of problems leading to linear equations.

5. Solution of problems continued. Lessons 4 and 5 include at least one problem of each of the standard types: rate of travel,

rate of working for two men, mixtures, levers, income on investments, etc.

6. Formulas. Definition, derivation, use, etc.

Special Products and Factoring

7. Multiplication of binomials, and factoring $ax^2 + bx + c$.

8. Factoring the difference of two squares, and quantities that can be written as the difference of two squares.

9. Prime Factors. Factoring by grouping.

10. The Factor theorem.

11. Review and drill on a miscellaneous list of quantities.

12. Solution of equations by factoring.

Fractions

13. The fundamental principle in reducing fractions. The laws for changing signs in the numerator, denominator, etc.

14. Multiplication and division of fractions.

15. Addition of fractions.

16. Complex fractions.

17. Review and drill on miscellaneous questions on fractions.

Fractional and Literal Equations

18. Selection of the proper multiplier or denominator after actorin g the denominators.

19. Equations containing decimals. Finding answer to a certain number of decimals.

20. Literal Equations. The difference between $ax = b$, $a + x = b$, $a + kx = b$ and $ax + bx = c$.

21. The meaning and solution of a "literal" equation, and problems containing "literal" numbers.

22, 23. Continuation of lesson 21. Further work on formulas.

Graphs

24. Review of elementary work on graphs. Since many pupils even to-day have never done any work on graphs even in the ninth grade, the subject must be presented thoroughly.

25. The functional notation, and graphs of functions.

26. Direct and inverse variation of variables.

Sets of Linear Equations

27. Review of the graphic solution of two linear equations. The graphic significance of dependent, independent, and inconsistent equations.

28. Review of the multiplication-addition and the substitution methods.

29. Continuation of lesson 28, with emphasis on literal equations.

30. Solution of a set of 3 linear equations, with problems.

31. Continuation of lesson 30.

32. Review and general drill. Expressing one variable in terms of another when several equations are given.

Radicals

33. Review of Fundamental Laws. Typical errors to be avoided.

34. Reduction and changes in the radicand.

35, 36. Addition, multiplication, and division.

37. Geometrical problems involving radicals.

Exponents

38. Review of Fundamental Laws. Meaning of Zero and Negative exponents.

39. Meaning of Fractional exponents.

40. Drill on multiplication, division, etc.

41. Equations in which x has a fractional exponent. Miscellaneous problems on exponents.

Logarithms

42. Definitions. Graph of $x = 10^y$.

43. Rules for finding the characteristic. Use of tables for finding the mantissa.

44. Interpolation in finding logarithms.

45. Interpolation in finding anti-logarithms.

46. Use of logarithms in multiplication.

47. Use of logarithms in division.

48. Use of logarithms to find powers and roots.

49. Drill on computations.

50. Use of logarithms in connection with various formulas, such as the area of a triangle in terms of its sides, etc.

51. Miscellaneous drill and review.

During the remainder of the semester several problems in computations are assigned each week so that the pupil may acquire facility and accuracy. This is a technique that is acquired only slowly.

Imaginaries

52. Definitions. Addition. Use of the symbol i.
53. Multiplication and division of complex numbers.

Quadratic Equations

54, 55. Review of the method of completing the square. Drill.
56. Problems leading to quadratic equations.
57. Literal quadratics. Derivation of the formula.
58. Study of the sum and product of the roots so that this method may be used to check solutions.
59, 60. Problems and equations solved by the formula.
61. Equations that can be reduced to the quadratic type.

Theory and Graphs of Quadratics

62. Character of the roots determined by the discriminant.
63. Graphs of $y = ax^2 + bx + c$.
64. Graphs of the fundamental types, as $y = x^2$; $x^2 + y^2 = r^2$; $x^2 - y^2 = a^2$.

Sets Involving Quadratic Equations

65. Solution of one linear and one quadratic equation.
66. Derivation of a linear equation when a linear equation is not given.
67, 68. Problems leading to sets of equations.
69, 70. Graphic solutions; their meaning and use.
71. General review of quadratics.

Radical Equations

72. General method of solving by squaring both members. Extraneous roots.
73. Drill on equations.
74, 75. General review on all types of equations and sets of equations that have been considered in the course.

Progressions

76. Definitions of arithmetic and geometric progressions. Finding d or r.

77. Arithmetic means, and the formula for the nth term in an A. P.

78. Formula for the sum of n terms of an A. P.

79. Problems dealing with arithmetic progressions.

80. Geometric means, and the formula for the nth term in a G. P.

81. Problems dealing with geometric progressions.

82. Formula for the sum of n terms of a G. P.

83, 84. Infinite geometric progressions; formula for the limit which such a progression approaches.

85. Review of both kinds of progressions.

Binomial Theorem

86. Statement of the theorem, and practise in writing the first five or six terms of such expressions as $(a + 2b)^{10}$.

87. Applications to compound interest, and the computation of 1.06^8 to three or four decimals.

88. Derivation of the first 4 terms of $(a + b)^{1/2}$ and $(a + b)^{1/3}$ and the use of the resulting formulas to compute square roots and cube roots.

89. What mathematical induction is.

Before we lament the omission from this outline of many of the objectives in Professor Reeve's list we should note that the latter was prepared "with the help of a large number of Teachers College students who are experienced teachers of mathematics." This explains why his list is such a complete one; doubtless each teacher who read the list added a few favorite topics, and I shall add another one shortly. My outline, on the other hand, was prepared in a different way. During the teaching of 13 classes in the subject I recorded each day what was accomplished in that day's work. My records show, for example, whether 15 minutes or 20 minutes were needed in class to discuss how the characteristic of a logarithm is determined; the records show how many minutes were needed to discuss why $x^0 = 1$, and how many different types of factoring were reviewed in a period of 40 minutes. To make sure that the class was not traveling too fast or too slow a 20-minute test was given once a week. Pupils who failed in the test were required to repeat the test (after a suitable interval for further study) as often as necessary until a satisfactory grade was reached. By noting the most frequent errors I could decide whether more or less time was needed for certain topics. The homework assigned each day was supposed to be sufficient to keep the pupil busy for 60 minutes. Every teacher knows what a great mistake it is to assign more work than can be discussed and corrected. Under ideal conditions the teacher would examine, correct, and return to the pupil every piece of work that the pupil does. Since this is impossible, time must be taken in class to discuss the errors so that the pupil will not continue making the same mistakes repeatedly. This discussion and presentation of problems in class restricts the number of topics that can be studied.

One objective which I believe ought to be included in this list and in the list of objectives for every high school subject is:

Ability to read a set of *printed* explanations; ability to study printed directions until they can be understood without *oral* guidance from a teacher.

In simple language this means that the pupil should learn to get along without a teacher. If the teacher, standing before the class, explains how to solve a certain type of problem and then requires the pupil to solve ten similar problems, the pupil is acquiring only a collection of mathematical facts. Even when the teacher explains a mode of attack for problems in general, the pupil is learning just that much and little else. I believe that even intermediate algebra, just as much as beginning algebra or history or physics, should be used to teach the pupil how to study how to work, how to attack a disagreeable task. The pupil should be assigned certain paragraphs in the book, taught how to study that topic until he has found its important items, can close the book, stand before the class, and tell what he has learned. This mode of treatment takes considerable time. Much time can be saved by having the teacher talk instead of letting the pupil talk. But if the only object is to save time why not have the teacher solve the problems in the book. If the teacher talks less and the pupils talk more, we shall need to eliminate still more of the objectives.

Frequently when listening to an eloquent speaker or when reading some article in the mathematical journals I have felt that certain other topics should be included in my outline, and I would reexamine my records. I allow, for example, five lessons (numbers 33 to 37) for Radicals. Can I reduce it to four? Or, can I reduce the time on Sets of Linear Equations from six lessons to five? When discussing such questions with other teachers I find that they invariably answer "Why not?" Then when I produce my records and ask "Which of these assignments shall I omit?" the answer is also invariably "Well, I guess you have already cut the time to the minimum." Until I find new ways of teaching I cannot include more topics.

I have not found time to teach the following topics (I mention them in the order in which they occur in the list of objectives):

Synthetic division.

Variation (except in so far as I can cover in a single period the difference between direct and inverse variation).

Determinants.

Write an equation that shall have given roots.

To know that imaginary roots always enter in pairs.

To recognize from its graph the general characteristics of a cubic equation in one variable.

To find the kth term of the expansion of a binomial. (There is a formula for the kth term; but if I wish to practise with formulas, I would prefer a more useful one such as finding the area of a triangle in terms of its sides).

The simplifications of the radicals in item 12 on page 157 should, I believe, all be done by writing the quantities with fractional exponents.

To change the index of a radical. (There is no need for this when fractional exponents are used).

To find the square root of such expressions as $\sqrt{13} - 3\sqrt{7}$.

To find the square root of a polynomial.

To find the square root of a number to two decimal places. (There is no need for this after the pupil has learned logarithms. And for training in technique there are other and better topics.)

To discover by means of a table of the powers of 2 how to solve such exercises as 16×64, $2048 \div 32$.

To make tables of the powers of other numbers than 2, and to solve problems by using these tables.

To apply logarithms in the work in trigonometry.

To use logarithms in solving simple exponential equations.

To interpret the graph of an imaginary number like $\sqrt{-1}$.

To graph complex numbers. (This topic is less important than many others. Also, the word "graph" is decidedly misleading when used with complex numbers. The pupil does not prepare a table showing the corresponding values of two variables when he draws the graph of $2 + 3i$.)

No required assignments are made on the history of algebra. This is optional reading. When the day's program is upset in some unusual way (the football team having defeated its bitter enemy) pupils talk briefly on this optional work.

One objection frequently made against intermediate algebra is that it is hardly more than a repetition or review of the ninth grade algebra with perhaps some more difficult problems of the same type. I have tried to avoid this objection (and Professor Reeve's list of objectives certainly avoids it) by devoting at least two thirds of the time to the really new topics. I have, in fact, been guided by two assumptions:

1. More than half of the pupils who take intermediate algebra will continue with the study of mathematics for some time. If they take the next semester's work, which at Hyde Park is trigonometry, they must have facility with logarithms, complex fractions, radicals, and literal equations. If the pupils go to college and take any required mathematics, these topics are again the ones in which skill is required.

2. The topics suitable for the pupils in the preceding group are just as suitable as any other topics for the pupils who do not expect to continue in mathematics but are taking intermediate algebra for some other reason. No one has yet proved that there is any more disciplinary value in one topic than in another.

It would be a great service to all if other teachers would discuss how many of the objectives in Professor Reeve's list can be attained in a definite amount of time.

MATHESIS

By ELLA BROWNELL,

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Scene

The scene is an underground studio or workshop. Incense is burning in a large black kettle hanging from a black tripod at front of stage. A red electric bulb under the kettle is surrounded by sticks of kindling wood. Two large portable blackboards are placed one at each side of front stage. Small table with white paper spread ornamented with black paper figures (circle, trapezoid, triangle, etc.) is concealed behind left blackboard. On the table are the following articles: cardboard cylinder, three ice cream cones filled with sand and having same base and height as cylinder. Two eighteen-inch boxes at right front stage covered with white paper and ornamented with black circles to imitate dice.

Characters

The boys wear black or dark trousers and white blouses; the girls, black skirts and white middies. Each pupil having a speaking part has large white card hung with black ribbon about the neck. On the card is drawn the proper symbol in large black type—*i.e.* Simple Interest has the percent sign (%) on his card. Plane Geometry (Π), etc. Characters to be written on cards are in parentheses.

Witch—Black dress and cape, tall pointed black hat, long hair made from unbraided rope and carries broom with long black handle.

Mr. Decimal Point (.)—black bow, white arrow.

Simple Interest (%).

Cutey Angle (V)—Black cap with long white plume.

Two Cone People—Carry ice cream cones filled with white cotton streaked with red ink.

Plane Geometry (Π)—Carries white hoop with black string tied across for diameter.

Isosceles Triangle (△)—Carries large white protractor.
Mr. L. W. H. ($v=lwh$)—Carries white measuring stick.
Miss Numerator—Denominator (5/8).
Miss Elementary Algebra (x^2y^3z).
Small Number (3).
Large Number (984,762,351).
Zero (0).
Infinity (∞).

As many more as stage will accommodate in order to have a crowd of interested lookers-on.

The Play

Music.[1]—Enter Witch, who advances slowly toward kettle and stirs contents with large black spoon while Decimal Point sneaks up from rear of stage imitating motions of witch. Music ceases.

Witch (much astonished): My little man, what are you doing down here in this cave of the underworld? How *dare* you come here?

Mr. Decimal Point: O witch, my name is Decimal Point and I'm not afraid of anything. I go *everywhere*. I crawl into all sorts of places whether I'm welcome or not. With my bow and arrow I can shoot anything I meet. You see I've been out hunting with my friends and when we discovered this cave they wished me to enter and inquire what you are doing and to whom this cave belongs.

Witch: This is the cave of Æolus, the king of the winds, and this is the isle of Æolia where Æolus and his six sons and six daughters live, keeping eternal carousal.

I, boy, am Hecate, a mysterious divinity. *I* represent the darkness and terrors of night. *I* haunt the crossroads and graveyards. *I* am the Goddess of Sorcery and Witchcraft. *I* am seen only by the dogs whose barking tells of my approach.

Mr. Decimal Point (stealing up to the kettle): May *I* ask what you are boiling in that kettle?

Witch: Tut, tut, my boy, *I* am boiling up answers.

Mr. Decimal Point: Boiling up answers, answers to what, pray tell.

[1] Kolling, Carl, Op. 147, No. 3. Flying Leaves. Century Music Co. 231–235 W. 40th St., New York City.

Witch: The king of the winds has two henchman, Boreas, the North Wind who blows directly north from our isle for 300 leagues until he reaches the City of Ignorance, and Eurus who blows directly east for 400 leagues until he reaches the City of Inaccuracy. King Æolus has commanded me to boil out the number of leagues that a third wind must blow in whirling cross country from the City of Ignorance to the City of Inaccuracy.

Mr. Decimal Point: O, witch, that's easy. Do you mind if I write on this wall? (Looks at blackboard at left where odd-shaped white marks have been made.) But what are all these marks?

Witch: Those are records of my evil deeds—all the crimes I have committed this week.

Mr. Decimal Point: Sorry to upset your bookkeeping, but let me show you how to find the number of leagues for the third wind. (Advances to board at left stage, erases it and illustrates his talk by the use of a large right triangle.) This is 300 leagues to the north and this the City of Ignorance, then 400 leagues to the east and here is the City of Inaccuracy. Now the square of 300 is 90,000; the square of 400 is 160,000. The sum of the two squares is 250,000 and the square root of 250,000 is 500. Now the third wind must blow 500 leagues according to the teachings of dear old Pythagoras.

Witch: And who is the *god* Pythagoras?

Mr. Decimal Point (with disgust): Pythagoras is not a *god.* He lived on the isle of Samos and he has taught us that in any right triangle the square of the hypotenuse is equal to the sum of the squares of the legs. Now, witch let me call in my friends who are waiting outside the cave, they can tell you a great many more interesting things.

(Witch sits down on large dice at right stage. Much noise and laughter off stage. Mr. Decimal Point goes to rear and beckons. Noise ceases, music.[2])

(Enter all other characters, also sufficient number of pupils to make stage well filled.)

Mr. Decimal Point: Now, Simple Interest, will you sing a mathematical song to our honored hostess of the underworld?

[2] Smith, Seymour. Dorothy—Old English Dance. Century Music Co.

Simple Interest: With great pleasure, Mr. Decimal Point. (Steps to front stage and sings the Number Song.[3])

Mr. Decimal Point: That's fine; thank you very much, Simple Interest. I didn't know that you could sing so well.

Cutey Angle (Skips forward in a rollicking fashion): Bet yer I know some mathematical sums that you can't answer. What's a tall coffee pot in use called? (All shake heads.) Give it up? Hypotenuse.[4]

If you should lose your parrot what would you say? (All shake heads.) Give it up? (With a giggle) Poly-gone.[4] What's an article for serving picnic ice cream?[4]

The Cones (coming forward eating their ice cream): Cones.

Cutey Angle: I must go out now to see the race.

Witch: What race?

Cutey Angle: Human race—don't worry, you don't belong to it.[5]

Mr. Decimal Point (to cones): What do you know about cones?

First Cone: A cone is formed by rotating a right triangle about one of its sides.

Second Cone (to First Cone): How many cubic inches of ice cream does your cone hold. (Borrows white measuring stick from Mr. L. W. H.) It is 4 inches high and has a radius of one inch, so it holds 1/3 of 3 1/7 times the square of one times 4 cubic inches or 4 1/5 cubic inches.

First Cone: For the land's sake, lucky you multiplied by 3 1/7 to make the answer larger. I don't feel as if I had eaten even 4 1/5 cubic inches of strawberry ice cream. Why not multiply by 19 7/8 or something to make us feel as if we had really had a good feed.

Mr. Decimal Point (in disgusted manner to Cutey Angle who has procured the Witch's broom and is sweeping the floor): What in the world are you trying to do with that broom, Cutey Angle?

Cutey Angle: I'm sweeping up the jokes that didn't get over the footlights.

Plane Geometry (anxiously coming to front stage holding up a white hoop): I can tell you about that 3 1/7. See my hoop.

[3] MATHEMATICS TEACHER, Vol. XVIII, October, 1925, page 353.
[4] MATHEMATICS TEACHER, Vol. XVIII, October, 1925, page 356.
[5] From Criss Cross as played by Fred Stone, New York City, 1927.

I have broken it so I tied it up with a string. This black string is the diameter and it is 17 inches long. When I multiply the length of the diameter by 3 1/7 I obtain 53 3/7 inches, which is the distance around my hoop. The Greeks used their letter Pi (points to the "Π" on his card) to represent 3 1/7. Pi is the Greek letter P, and stood for "periphery" which means "circumference." Many values have been used for Pi, ranging from 3 to the value out to 707 decimal places. At present every school boy knows it as 3.1416 or about 3 1/7.

First Cone: I may be dumb, but I don't see yet what the 3 1/7 and the length of the circumference has to do with my ice cream. I don't eat the cone or the rim around the top. I eat what's inside.

Plane Geometry: Well, let me have the floor once more, and I'll explain again. Before you can find the volume of your ice cream . . .

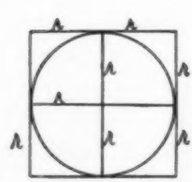
Cutey Angle (breaking in from the right side): It will all be melted.

Plane Geometry: Before you can find the volume of your strawberry ice cream you have to find the area of its base which would be a circle.

Cutey Angle (with a giggle): Ice cream with a "*base,*" strawberries with a "*sporano*"—Ha, ha, ha.

Mr. Decimal Point: Silence please; no more nonsense from you, Cutey Angle. Continue now, Plane Geometry, please.

Plane Geometry (steps to large blackboard at right): Here's your circle with radius called "r." How many squares are there?



Whole Class (witch looking on): Four squares.

Plane Geometry: What is the area of one square?

Whole Class: *r* square.

Plane Geometry: What is the area of all four squares together?

Whole Class: Four *r* square ($4r^2$).

Plane Geometry: Do you think the area of the circle is more than the area of the four squares?

Whole Class: No, the circle area is a little less than the area of the four squares.

Plane Geometry: Well, it is $3\ 1/7r^2$ and not quite $4r^2$. So the area of every circle is found by squaring the radius and then multiplying by 3 1/7. Now First Cone do you understand that brother second cone was right when he told you to square one and then multiply by 3 1/7. Then the cream is 4 inches deep so you have to take that into account, and your cone (borrows the cone from Second Cone) is pointed and holds just 1/3 as much as a cylinder, provided they have the same sized bases and equal heights. Let me show you. (Motions to two pupils who bring out small table from behind left blackboard and place it at front stage.) See this cylinder. Its base has the same area as the base of this cone. (Compares the two bases carefully.) The cylinder is empty, but each of these three cones is filled with sand. Watch me. (Empties sand from three cones into cylinder slowly so all can see. Witch appears to be quite overcome.) It takes three cones full to fill the cylinder, therefore one cone holds just 1/3 as much as the cylinder provided the dimensions are the same.

The formula for the volume of a cone is $V = 1/3\ bh$.

Isosceles Triangle (who has been measuring the angles which the tripod made with the floor by using the large protractor): Just see, folks, what I've found. These sticks form an isosceles triangle with the floor of the cave as a base. Each base angle measures 55 degrees, and since there are 180 degrees in every triangle, there must be 70 degrees in the angle up here. (Measures the vertex angle above the kettle.) Yes, there are 70 degrees in the vertex angle.

Mr. L.W.H.: Pardon me, Witch, but are you aware that the white box upon which you are sitting is a cube. What use do you make of the dots on it?

Witch: I toss these dice and count the black spots on them. They tell me how many *evil deeds* I must do each night.

Mr. L.W.H.: O Witch, don't be so cruel, let me explain what wonderful mathematical forms your dice are. The length, width, and height are all equal. (Measures carefully with white measuring stick.) It is just 18 inches on an edge, so its volume will be 18x18x18 cubic inches or 5,832 cubic inches.

Mr. Decimal Point: Is our poetess, Miss Numerator Denominator, present?

Miss Numerator Denominator: Yes, Mr. Decimal Point.

Mr. Decimal Point: Will you recite one of your latest poems for us—the one you composed while you were on your vacation at Mount Quadratic last summer with your friend Miss Elementary Algebra.

Miss Numerator Denominator: With great pleasure Mr. Decimal Point. (Comes to front stage and recites poem entitled First Aid in Algebraic Fractions.[6]

Mr. Decimal Point: We could tell you many more interesting facts, old Witch, but now that you have learned that it is 500 leagues from the City of Ignorance to the City of Inaccuracy, I hope your boss, old King Æolus, will not scold you for spending a little of your time with us. We must all return to the upper world now and go back to our duties at our beloved - - - - School. In parting, we'll sing you a song. (Entire group forms acute angle on stage with vertex at rear and sings the Number Song.[7] (Pupils point to the cards of the following as they are mentioned in the song: "numbers great," "numbers small," "vast infinity," "zero."

Exit All

[6] Mathematics Teacher, Vol. XIX, February, 1926, page 101.
[7] Mathematics Teacher, Vol. XVIII, October, 1925, page 353.

SOME PEDAGOGICAL ASPECTS OF GEOMETRY TEACHING

By MABEL SYKES

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Oftentimes things are said that are so fundamentally true that they never grow old.

Many years ago, Dr. J. M. Coulter, the botanist, read a paper on the "Mission of Science in Education." Something that he said in this paper has a direct application to much of our geometry teaching today. He contrasted two types of instruction in the following words: "In the one case, the facts are presented in the helter-skelter fashion, solid and substantial enough, but a regular mob, with no logical arrangement, no evolution of a controlling idea. Details are endless, no emphasis brings out certain things into prominence and subordinates others, and the whole subject is as featureless as a plain, where the dead level of monotony kills off every one but the drudge. . . . In the other case fewer facts are presented, but they are the important ones, and marshalled in orderly array, battalion by battalion. They move as a great whole towards some definite object. . . . Instead of a level plain, there are mountain peaks and valleys. There is a perspective: there are vistas from every point of view."

Just at this point it is of the utmost importance that there be no misunderstanding. The subject matter in geometry texts is divided into chapters or books, and the theorems are so arranged that each proof depends logically upon what precedes. This is true and always has been true of all geometry texts; it is fundamental and essential to all geometry teaching. A more complete classification than this is suggested by the above quotation. Consider almost any chapter in almost any geometry text with the following questions in mind: Is there a controlling idea in this chapter? Should there be such a controlling idea? Are there several such ideas? If so what are they? Has each controlling idea its fundamental theorems or definitions, and are the other theorems and exercises grouped about these fundamental facts and made subordinate to them? In other words does the careful classification of material extend beyond the mere division into chapters, down to the last detail? Is this organization so clear and evident that the pupil is conscious of it? If the material in our geometry texts is not organized in this way, should it be and can it be so organized? To what extent do we use such organization in our work?

Some years ago, Dr. E. R. Hedrick at that time in the University of Missouri, said in a geometry report "It is a fundamental characteristic of the human mind that any lasting impression of a vast field requires distinctions in emphasis." In his syllabus of geometry theorems in the same report, theorems were printed in different kinds and sizes of type according to what seemed to him their relative importance. The most important theorems were printed in black type, the least important in very small type, with the others arranged in between.

If one may judge from the textbooks that have appeared since this report, this feature of the report made little or no impression. Here again a misunderstanding would be fatal. Surely, Dr. Hedrick never intended that different sizes and styles of type be used in printing theorems in geometry texts. It spoils completely the appearance of the book and accomplished nothing. What Dr. Hedrick suggested in regard to geometry teaching is exactly what Dr. Coulter said about all science teaching.

This brings us back to the arrangement of the subject matter within the chapters and to the necessity of stressing fundamental theorems and definitions by frequent use. Such an arrangement of material by careful classification is in accord with the best psychology and pedagogy as is suggested by the quotation from Dr. Hedrick given above. Dr. C. H. Judd, in his "Psychology of High School Subjects," page 62, says that "the only way to teach a student to solve originals is to teach him to analyze a new problem." Surely teaching pupils to solve originals is one of the skills that every geometry teacher hopes her class may acquire. No teacher to-day is satisfied with the "inferior training" that many of the class get by merely following the text, or by listening to the solution of exercises presented by the favored few who can rely upon inspiration.

Of course the element of inspiration can never be completely eliminated from the study of original proofs: nor is it desirable that it should be. Most pupils, however, need something besides inspiration in working on originals. This something is supplied by careful training in analysis. If the work has been so presented that certain theorems, definitions, and methods stand out in the mind of the student as fundamental, he can classify at sight any exercise that is within reason. He has then the first step in the analysis of the proof for which he is searching. To make an analysis, one must ask one's self a series of questions. One should start with what is to be proved, and by means of these questions unravel the proof step by step. If the geometry work is divided into a series of units, each clear cut and complete in itself, if each unit has its fundamental theorems or definitions with the rest of the work grouped about them, then all the pupils who deserve their credit in geometry can be expected to do easy originals.

The attention of the teaching force has recently been called to topic teaching, by Dr. H. C. Morrison of the University of Chicago. There are two principal objectives in geometry teaching: (1) to give the pupil an understanding of certain space relations and (2) to train him to make use of the facts presented. Both objectives can be attained most readily by organizing the work into units. However, each unit should not only be complete in itself, but should be small enough for beginners to grasp as a whole. This is not accomplished by making the unit so comprehensive that it includes an entire chapter or book, or by including at random the next five, ten, or fifteen theorems. Moreover, if a small unit is used, a concise preview can be given the class before the detailed study of the unit is begun. This preview affords the teacher an opportunity to give the class an understanding of the space relations involved in the unit. The training in problem solving should come in the more detailed and intensive study.

In what is usually called Book I, the most important idea is, of course, the use of congruent triangles. Probably very many, if not most, geometry pupils, do get this idea and get it so truly and thoroughly, that when they wish to prove segments or angles equal, they naturally look first for a pair of congruent triangles. But teachers should consider seriously whether this is because of the organization of the work or in spite of it. It might be noted that in presenting the subject of congruent triangles, it is not necessary to include in the unit all of the tests for such triangles. The first three tests are all that are really necessary to develop in the class skill in the use of such triangles. The other theorems can be introduced later where they seem to fit in best.

Besides congruent triangles, Book I usually includes the following topics:

I. Parallels and angles,
II. The sum of the angles of a triangle or polygon,
III. Parallelograms,
IV. Inequalities.

Opinions may differ as to what are the fundamental definitions or theorems in any particular topic. In fact, what these are, is not nearly so important as that there be such fundamental theorems or definitions, and that the rest of the material included in the topic be subordinated to what is fundamental.

In the case of parallelograms there can be little question. The definition is the fundamental thing. The work included can and should be arranged under the following heads: (1) properties of parallelograms, (2) congruence of parallelograms, and (3) tests for parallelograms. The entire subject should be so organized and taught that the pupil naturally makes proper use of the definition of parallelograms. In fact the only theorem that does not naturally depend directly upon this definition is *The diagonals of a parallelogram bisect each other.* After the essential theorems included in the unit have been studied, the class must be trained in the application of the tests for parallelograms by the use of numerous exercises. If the work is arranged in this way, pupils will not only have a clear and lasting impression of the important facts about parallelograms, but will be able to attack or analyze reasonably hard exercises involving parallelograms. The importance of the definitions as fundamental can and should be emphasized in discussing the properties of special parallelograms and of related quadrilaterals, such as the isosceles trapezoid and the like.

In presenting the subject of parallels and angles, three groups of theorems should be included: (1) tests for parallels, (2) angles made by parallels and transversals, (3) supplementary theorems. In this study the definition of parallel lines and the assumption, *Through a given point, one parallel and only one, can be drawn to a given line,* are, of course, fundamental. In building upon these basic facts, there is room for considerable difference of opinion as to the arrangement of material. If the theorem: *If two straight lines in the same plane are cut by a third straight line so that the alternate interior angles are equal, the lines are parallel,* and its converse be taken as the fundamental theorems for the first two groups, two desirable results may be secured: (1) the other theorems included in the unit are easily grouped about these two, and (2) the pupil has at his command a test for parallels which can be applied in the majority of early exercises in which it is necessary to prove two lines parallel. The use of the two theorems involving perpendiculars and parallels, *Two lines perpendicular to the same line are parallel,* and *A line perpendicular to one of two parallels is perpendicular to the other,* may be made the subject of a special lesson or two by the proper choice of exercises.

When one considers that the training of pupils in problem solving, is one of the main objectives in geometry teaching, it is a little difficult to see why the sum of the angles of a polygon should not follow immediately the theorem concerning the sum of the angles of a triangle. Yet it is rarely so placed in textbooks.

In studying circles, three sub-topics stand out in one's mind as of first importance, namely: (1) circles and chords, (2) circles and tangents, (3) circles and angles. These three topics are capable of definite, clear cut unit treatment. The pupil is entitled to a working knowledge of two important facts in this connection: (1) that in dealing with chords, one thinks first of the diameter perpendicular to the chord, and (2) that in dealing with tangents, one thinks first of the diameter drawn to the point of contact. This working knowledge is only obtained by stressing the corresponding theorems by proper arrangement of other theorems and exercises. In studying the measurement of angles, a very interesting unit may be worked up, making everything subordinate to the theorems concerning the measure of an inscribed angle and the measure of an angle formed by a tangent and a chord.

Work on loci is a separate topic and should be so treated.

The topics that are generally treated with the least regard to the laws of pedagogy, are those involving proportion and similarity. One proof of this fact is the difficulty that a solid geometry class usually has with the theorem concerning a plane parallel to the base of a pyramid. If the pupil has studied similar figures in plane geometry so that the fundamental facts and methods have made a lasting impression, he can analyze and prove this theorem with very little assistance. But to do this, he must know that to prove two polygons similar, he must prove their angles equal and their corresponding sides proportional, and that to prove two ratios equal, he has certain well defined methods. In this case, he has to prove each ratio equal to a third ratio by means of similar triangles. Does the pupil know these facts? He certainly will not be conscious of the methods of proving ratios equal, unless his attention has been called to them, and he has had drill in their application. This drill includes the use of the theorems concerning ratios of segments made by parallels and transversals and the use of the first test for similar triangles. He must also be taught to recognize the case in which it is necessary to prove two ratios equal to a third.

The properties of similar figures are listed by the Society for the Promotion of Engineering Education, in their "Syllabus of Mathematics" as among the "facts that a student should have so firmly fixed in his memory that he will never think of looking them up in a book." It is difficult to see, however, how he is going to learn the properties of similar figures if they are scattered about over two chapters in his text book, and if his attention is never called to them by any summary, written or oral. This topic concerning the properties of similar figures makes a good unit for presentation if it is postponed until after areas are studied. Then the theorem concerning the ratio of the areas of two similar figures can be included as indeed it should be. The entire group of theorems can then be stressed by the use of numerical exercises.

The topic involving the theorems concerning the areas of plane figures is in general well organized, and a large number of numerical exercises are generally available. Whether or not it is desirable to train pupils to recognize polygons that can be proved equivalent, is a matter worth considering. If it is desirable to include this topic a very careful organization of material is necessary, for very little consideration beyond a few scattered exercises is given to the matter in most texts.

The list of topics here mentioned does not claim to be exhaustive; it is suggestive merely. Many recent texts call attention to the plan of proof before giving the proof in detail. This adds greatly to the intelligent reading of text proofs. Many recent texts also give more or less attention to analysis as a method of attacking originals. But no adequate training in analysis can be given without a careful organization of material into units with proper placing of emphasis. The quotations from Dr. Coulter and Dr. Hedrick given at the beginning of this article, may seem like ancient history, but good pedagogy is never out of date and the whole matter has been revived very recently by Dr. Morrison's insistance on topic teaching, and still more recently by a statement in the article by Miss Gertrude E. Allen, in the Second Year Book of the National Council of Teachers of Mathematics. On page 260 in discussing the Outline of the Course in Geometry, she says, "Regardless of the text, it is essential that the teacher *emphasize the relative importance of the significant theorems, group related theorems,* and *generalize closely related groups* in one theorem when practical.

NEW BOOKS

An Introduction to Mathematical Analysis. By FRANK LOXLEY GRIFFIN. Houghton Mifflin Co., N. Y. City, 1921.

Freshman Mathematics. By GEORGE WALKER MULLINS, and DAVID EUGENE SMITH. Ginn and Co., N. Y. City, 1927.

What mathematics should follow the courses given in the ordinary high school? Present opinion is crystallizing on four points: (1) There seems to be an agreement that the subsequent work in mathematics should touch life at more points than has hitherto been the case with our formal courses in trigonometry, college algebra, and analytics; and that they should provide problems that inculcate an appreciation of the power and place of mathematics in building up the civilization of which we are the beneficiaries. (2) As to the manner of presentation, there is a growing conviction that the element of discovery must be given a larger place, even at the expense of a less rigorous and systematic exposition in the initial stages. The good teacher has always been guided by this principle, but it is only recently that textbook writers have made this one of their cardinal practices.

(3) As to arrangement of material the so-called spiral method, sometimes much abused, but more often successful has come to the fore in secondary school mathematics and is fast gaining favor in college courses. Whether given in unified courses or as separate subjects it is realized that the more difficult features of certain topics cannot be mastered until a broader basis has been laid. (4) Another tendency, closely related to the spiral arrangement and practically an extension of it, relates to the organization of the curriculum of college mathematics. In secondary school mathematics it has been observed that more interest and better results follow by omitting the more abstruse topics of algebra and geometry and inserting the simpler elements of numerical trigonometry, determinants, the graphical solution of equations, etc.,—subjects hitherto thought of as college disciplines. The new curriculum harmonizes better with the student's capacity, adapts itself better to his experience,

and forms a better basis for courses in industry and science which he may take up afterwards. Many educators feel that the same obtains for college mathematics; and some teachers and textbook writers have acted on this conviction. This progressive curriculum has found expression in so-called general or survey courses for freshmen. These courses include, besides trigonometry and algebra and analytics, the elementary concepts of the calculus. In the execution of this plan, however, there develops a somewhat sharp division: one wing, hitherto the more aggressive, advocates that this general course be built up as unified mathematics; the other desires the different fields in the main to be kept apart, though closely correlated.

Consonant with these progressive principles a number of new texts have been published in the last decade. But your reviewer knows of no other book that has so successfully met all these desiderata as the two listed above.

Griffin's *Introduction to Mathematical Analysis* is a most skillfully graded and carefully unified course in college algebra, trigonometry, analytics, and the simpler processes of the differential and integral calculus. The book is a veritable thesaurus of interesting problems, so that even a person who teaches from conventional texts will do well to have Griffin on his desk for problem material. This text is well-nigh unique in that the student learns nearly all the routine and technique by solving actual problems from physics, mechanics, statistics, finance, and engineering. The book was not put on the market hurriedly: there were too many derelicts on the mathematical shores for Griffin, a believer in unified texts, to wish to add one more failure. So for nine years he tried out the material with his freshman classes in Reed College, eliminating problems and methods that proved unsatisfactory.

This work covers 512 pages, under 15 chapter headings. Chapter I, entitled "Functions and Graphs," is a classic. The reviewer has never seen a finer bit of pedagogy in any language than this exposition of the graph and its uses. In learning to plot, much of the student's time is necessarily consumed with mechanical routine. What a pity, then, that all the reward most books on algebra and analytics can offer a student for his labor is a mild study of functionality and its application to interpolation, while the interesting topics of rates and summations are deferred to a later course! Through a list of "fundamental problems of variation," Griffin introduces the student graphically to the concepts of differentiation and integration as well as interpolation. This chapter is really an epitome of the whole book.

Chapters II, III, IV take up analytically the concept of limit and the processes of differentiation and integration of polynomials. It is the expert teacher who has written these chapters; and the reviewer can testify from actual experience with the text that students who have been introduced to the limit concept through these chapters, though they may fumble on technique in their sophomore calculus, they really *know* what a derivative is and what constitutes an integration. Formal trigonometry and logarithms are taken up in Chapters V and VI. This is the least satisfactory part of the "Analysis." The smooth progress of the first chapters seems interrupted. The insertion of the trigonometric functions at this place appears a little forced. One wonders whether formal trigonometry, any more than demonstrative geometry, incorporates easily into a unified course. Chapter VII, on Logarithms and Exponential Functions, is another gem. One will have to search far before one finds a more pedagogic development of the natural logarithm system and a more "natural" account of the "natural base" e. Griffin's use of the logarithmic and semi-logarithmic graphs to discover scientific laws, while unusual in freshman texts, is worthy of imitation by textbook writers, considering that the processes are not involved and the applications possible in daily life are quite numerous.

The chapters on coordernates, trigonometric analysis, solution of equations, complex numbers, and the definite integrals are taken up more in the conventional way. We feel that some of these chapters are too crowded. It would have added to the value of the "Analysis" to have had less work in differentiating trigonometric, logarithmic and exponential functions and less complicated work in the definite integral; especially so as more time should have been given to the conic sections.

Two other chapters deserve special mention. Chapter XIII teaches Progressions and Series by showing their use in the Theory of Investment and the discovery of scientific laws. Chapter XIV, on Permutations, Combinations, and Probability, gives the rudiments of Statistical Method in a way that should be of help to the student in his later work or reading; this work should prove especially valuable to the student who takes no more formal mathematics.

Freshman Mathematics, by Mullins and Smith, is not a unified course in the strict sense of the word; for the different fields of mathematics, though closely correlated, are in the main treated separately. The list of chapter headings gives an idea of the book: I. Elementary Algebra Applied. II. The Binomial Theorem and Series. III. Logarithms. IV. Trigonometry. V. Analytic Geometry. VI. The Calculus. VII. Numerical Equations. VIII. Practical Measurements. In this work the authors aim to present the basis for a one-year course that will "open the door of college mathematics to all by showing the meaning and the purpose of the science and its general usefulness in the various fields of intellectual activity."

This text is admirably adapted for classes having had only two semesters of algebra. Classes that have had three semesters should study the first three chapters eclectically. But even where the subject matter may be elementary, the entire treatment both as to its theoretical aspect and as to its practical application has in mind the interests and point of view of the college student. To quote from the preface: "Instead of beginning with the conventional formal review of algebra, the plan has been adopted of setting forth clearly the types of work in which the student is likely to need algebra in his subsequent study, and following this with a review that is limited strictly to the essentials of the subject. By being based upon the principle of subsequent usefulness, this review can be placed upon a somewhat higher plane than is possible with the conventional type."

Of special merit is the chapter on trigonometry. Trigonometry should appeal to our common sense and everyday experience more than most pre-calculus studies; but our texts generally clutter up the main principles and processes with digressions and a welter of phrases. The contrast in Mullins' and Smith's text is refreshing. The exposition is simple and attractive and the subject matter foundational. The plan of the book permits the authors to take up logarithms as a coordinate topic between the chapters on algebra and on trigonometry and this partly assists in simplifying the discussion of formal trigonometry. Too often the student has to unlearn the idea that logarithms is a particular branch of trigonometry; but in "Freshman Mathematics" he sees the individual status of the subject of logarithms as well as its application to trigonometric processes. One could wish, however, a little fuller treatment of the section the authors choose to call Analytic Trigonometry.

In line with recent opinion the authors have inserted a chapter on the calculus. And they have called it "Calculus." The reader may remember Downey's "Higher Algebra," extensively used in colleges twenty-five years ago. In this text was found considerable work in differentiation, in an algebraic setting. But the calculus was there. Later the work in differentiation was omitted from our college algebras. It was an indefensible omission, and one rejoices to see the derivative again incorporated into our Freshman mathematics. Mullins and Smith confine themselves, and wisely so, to the calculus of the polynomial. The principal topics are rates and slopes, together with their applications to maxima and minima, and the integrations for areas and work.

The classification and organization of the material in this book is a work of art. The goal of the authors is so clear, the progress towards it so undeviating, the style of presentation so smooth, that one seems to be reading a poem. One is seized by a strong desire to try out the book before the living forms in a class room.

High schools planning to give advanced elective courses as recommended by the National Committee on Mathematical Requirements should examine these two books. Chapter I–IV (155 pages) of Griffin are well suited for a semester's course in elementary calculus; the writer knows of at least one high school that gives such a course, based on Griffin. The first four chapters of Mullins and Smith (161 pages), would form a good, practical course in logarithm and trigonometry for high school students.

The authors of "An Introduction to Mathematical Analysis" and "Freshman Mathematics" have adopted what is best in the European curriculum of freshman grade and have incorporated in their texts a judicious amount of proposed American reforms. They have thereby contributed a distinct service to the pedagogy of mathematics.

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SOLID GEOMETRY VERSUS ADVANCED ALGEBRA

By W. F. BABCOCK,

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The heads of mathematics departments in several Eastern colleges were asked the following questions:

> In your opinion if circumstances make it impossible to offer both solid geometry and advanced algebra in a coeducational secondary school course, which of the two is of greater value:
>
> (*A*) To the student who intends to continue the study of mathematics in college?
>
> (*B*) To the student who will take as little mathematics as possible in college?
>
> (*C*) To the student who is not going to college?
>
> In other words in a coeducational school that prepares almost all of its students for college would you advise continuing to instruct solid geometry and trigonometry in the senior year or substitute advanced algebra with an elementary treatment of the beginning of the calculus and trigonometry?

Twenty replies were received as listed below:

> In answer to *A*, 10 voted for solid geometry, 9 for algebra and one either.
>
> In answer to *B*, 11 voted for solid geometry, 6 for algebra and three for either.
>
> In answer to *C*, 13 voted for solid geometry, 6 for algebra and one either.

Extracts from the replies show as great a difference of opinion as in many other questions of pedagogy. There are two criticisms which seem fairly general: the first, that the college freshman is weak in elementary algebraic technique, and the second, that the introduction of the calculus into all secondary schools is questionable.

"I think that the difference is mainly one of teaching synthetic reasoning as in Euclidean geometry instead of teaching analytic reasoning which is the basis of higher algebra. I doubt if a student younger than 18 years ever understands analytic reasoning. I doubt the wisdom of beginning calculus then. We find that those who have had a little calculus are no better off after a month than those who have had no calculus before coming to college."

"If a student is going to a higher institution where mathematics is fundamental, the aim of the school preparation should be in the direction of maximum thoroughness in elementary work with a good deal of drill rather than the introduction of calculus notions with the attendant risk that the student may commence the college work with overconfidence."

"In regard to *B* and *C*, I would recommend taking neither."

"I certainly would not give even elements of calculus in a preparatory school."

"Advanced Algebra is largely preparatory to further study in mathematics and while it has some cultural value (training in logical thinking) it is not nearly so cultural as solid geometry which gives splendid opportunity to drive home the conception of a mathematical proof. The introduction of differentiation and integration of a polynomial into advanced algebra strengthens the course. Nevertheless, I believe solid geometry affords more educational opportunities."

"The calculus would be better left until later."

"If your pupils have enough algebra to handle solid geometry and trigonometry, give them calculus. I regard advanced algebra as preparation for the realities of mathematics. I believe the study of algebra should be supplanted by applications to trigonometry and the calculus as soon as possible. The tendency in colleges now is to regard the calculus as the goal to be reached as soon as possible."

"We find the average freshman very deficient in the ability to do transformations and solvings of elementary algebra so that here we find a desire to see more emphasis on the algebra. The usual high school course in trigonometry is, from our point of view, so deficient in analytical trigonometry that we find it necessary to repeat the subject for all who go on in sophomore analytics and calculus. Until the time comes when the 'analysis' is more generally taught, I do not favor the beginning of calculus. Theoretically it is fine, but there is much against it."

"Whether the pupil goes to college or not, a thorough working knowledge of algebra and trigonometry is far more useful than solid geometry. You could make no worse mistake than to waste valuable time, which ought to be used in giving the pupil a thorough working knowledge of fundamental elementary things, by attempting to give him a superficial acquaintance with advanced subjects for the comprehension of which he has, in general, neither the maturity nor the background."

ANNOUNCEMENT

Owing to a mistake in the recent advertising material sent out by the Mathematics Teacher some of our readers have the idea that we issue a September number. We are sorry that such a mistake was made, as no numbers ever appear in June, July, August or September.

Again, we are sorry that the large increase in membership to the National Council this autumn has made it impossible for us to get the October and November numbers to some of our new subscribers. In all such cases we shall begin with the December number.

Members of the Council will no doubt be interested in the mailing list by states which was used in sending out the November issue of the Mathematics Teacher. We hope to give a much longer list in a short time, but the following list will indicate to our present membership something of the nation-wide interest in the work which the Council is doing:

Alabama	65	Maryland	79	Oregon	27
Arizona	9	Massachusetts	235	Pennsylvania	279
Arkansas	23	Michigan	108	Rhode Island	17
California	145	Minnesota	70	South Carolina	21
Colorado	46	Mississippi	27	South Dakota	17
Connecticut	69	Missouri	58	Tennessee	20
Delaware	3	Montana	9	Texas	136
Florida	24	Nebraska	52	Utah	5
Georgia	39	North Carolina	70	Vermont	14
Idaho	12	North Dakota	10	Virginia	38
Illinois	302	Nevada	2	Washington	25
Indiana	125	New Hampshire	19	West Virginia	28
Iowa	89	New Jersey	110	Wisconsin	102
Kansas	136	New Mexico	7	Wyoming	11
Kentucky	30	New York	356	Philippine Islands	12
Louisiana	39	Ohio	165	Foreign Countries	96
Maine	25	Oklahoma	49		

In our next report we hope to see a large increase in the numbers from each state. All members who desire to assist us in our campaign for new members should write to the Mathematics Teacher, 525 W. 120th Street, New York City, New York, for advertising material, or send in subscriptions to that address.